A Note on Periodic Solutions to Flow in a Liquid Film

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During the past 25 years considerable attention has been given to the Kapitza method (1948) for the solution of the problem of falling film flow in the presence of capillary waves. It is shown that such methods can be classified as special cases of a general treatment due to Shkadov (1967). The limitations of each method are then readily understood. Dukler (1971) has demonstrated that the method is limited to low Reynolds number flows and thus the overall contribution of each method is made clear.

The work of Kapitza was in two parts. The first, a linear treatment, was concerned with the fundamental frequency alone in the periodic solution for velocity and film thickness, while the later nonlinear treatment added first harmonic terms. Kapitza neglected an important term $\left(v\dfrac{\partial u}{\partial y}\right)$ in his derivation, but in this discussion the corrected version is referred to as the Kapitza result.

Massot et al. (1966) effected a solution similar to the linear Kapitza case, but including all terms of the Navier-Stokes equations. However, an error was made in the boundary conditions so that their result does not correspond exactly to that ascribed to them below.

Finally, Shkadov used a systematic method for establishing the coefficients of a Fourier expansion in film thickness that does not necessitate neglect of nonlinear terms in the derivation. The results of Kapitza and of other workers are special cases of this method and it is the purpose of this note to make the relationship clear.

METHOD

A detailed description of the method of solution has been given by Webb (1970). The two dimensional Navier-Stokes equations, written in the x and y directions (downstream and normal to the solid wall respectively), and the equation of continuity must be solved with the boundary conditions.

(i) The no slip condition at the wall

(ii) The kinematic surface condition that expresses the fact that the surface is a streamline.

(iii) The normal and shear stress relations written at the interface including the effect of surface tension.

The problem is simplified by assuming that the wavelength is considerably greater than the mean film thickness and that the waves travel unchanged. The first assumption, taken with the y directional Navier-Stokes equation, shows that the pressure is constant over the film thickness. The latter assumption allows the definition of a coordinate system propagating with the wave velocity so that independent variables time and axial distance can be combined in a single variable $\xi = 2\pi(x - ct)/\lambda$. Further the velocity profile is assumed to be parabolic, which is true for parallel flows but neglects accelerational terms in this non parallel flow. The velocity profile is

$$u = 3\,\bar{u}\,(\xi)\left[\frac{y}{h(\xi)} - \frac{y^2}{2h(\xi)^2}\right] \tag{1}$$

The continuity equation allows the evaluation of the y directional velocity component

$$v = v\left(\bar{u}, \frac{\partial \bar{u}}{\partial \xi}, \frac{\partial h}{\partial \xi}, y, h\right) \tag{2}$$

When these relationships are substituted into the x directional Navier-Stokes equation, the result is an equation in two dependent and two independent variables. A new dependent variable ϕ is defined by the Fourier expansion.

$$\phi = \rho\sin\xi + \phi_{20}\rho^2\sin2\xi + \phi_{21}\rho^2\cos2\xi + \tag{3}$$

The film thickness can be expressed directly as a function of ϕ while the mean velocity can be related to ϕ through the continuity of mass relationship.

$$h = h_0(1 + \phi) \tag{4}$$

$$\bar{u} = (c\phi + u_0)/(1 + \phi) \tag{5}$$

Further an integral method is used in which y dependence is eliminated by considering mean values averaged over the film thickness. The result is the equation $\phi(\xi) = 0$.

$$\begin{aligned}
&\frac{\delta_L n^3}{h_0^2}(1 + \phi^3)\frac{d^3\phi}{d\xi^3} + \frac{3n^2}{h_0^2}\left[\nu\left(c - \frac{3}{2}u_0\right)\right.\\
&\qquad\left. + \frac{\nu}{2}(c - 3u_0)\,\phi - \nu c\frac{\phi^2}{2}\right]\frac{d^2\phi}{d\xi^2}\\
&\qquad + \frac{n}{h_0}\left[\left(c^2 - \frac{12}{5}cu_0 + \frac{6u_0^2}{5}\right) - \frac{2c^2}{5}\phi\right.\\
&\qquad\left. + \frac{c^2}{5}\phi^2 + \frac{9\nu n}{h_0}(u_0 - c)\frac{d\phi}{d\xi}\right]\frac{d\phi}{d\xi}\\
&\qquad + \left[\left(3g - \frac{3\nu c}{h_0^2}\right) + 3g\phi + g\phi^2\right]\phi\\
&\qquad + \left[g - \frac{3\nu u_0}{h_0^2}\right] = 0
\end{aligned} \tag{6}$$

A solution of (6) is sought in the form of the Fourier expansion (3), the degree of approximation being improved by the addition of higher harmonic terms. The multiple angle terms in the Fourier expansion are expressed in terms of the single angle and a function Q_i defined by

$$Q_i = Q_{i0}\sin^i\xi + Q_{i1}\sin^{i-1}\xi\cos\xi + \qquad + Q_{ii}\cos^i\xi \tag{7}$$

so that Equation (6) may be written

$$\sum Q_i = 0 \tag{8}$$

The problem is to evaluate the $(2k + 1)$ coefficients in

$$\sum Q_i = \sum_{i=0}^{i=k} M_{i0}\sin(i\xi) + M_{i1}\cos(i\xi) + R_{k+1} \tag{9}$$

The solution in the kth approximation is then obtained by setting the $(2k + 1)$ coefficients $M_{ij} = 0$. Thus each additional harmonic in (3) gives two further equations for the added coefficients ϕ_{k0} and ϕ_{k1}. The important point to notice is that the $M_{k-1}, M_{k-2}, \ldots, M_0$ contain a contribution from the $Q_{k+1}, Q_k, \ldots, Q_2$ and it is this contribution that Kapitza and Massot et al. have neglected. Taken with neglect of nonlinear terms in Equation (6) the simplicity of their results is understandable.

SOLUTION

Zero'th Approximation

$$\phi = 0 \tag{10}$$

$$\beta^3 = 1 \tag{11}$$

where $\beta^3 = gh_0^2/3\nu u_0$. Thus, the trivial result is exactly the Nusselt Equation.

First Approximation

$$\phi = \rho \sin\xi \tag{12}$$

The results obtained by setting the relevant $M_{ij} = 0$ are

$$M_0 = 0; \quad \beta^3 = 1 \Big/ \left(1 + \frac{3}{2}\rho^2\right) \tag{13}$$

$$M_{10} = 0; \quad \alpha = 3\beta^3 \tag{14a}$$

$$Re = 4s \left[\frac{\beta(3\beta^3 - \alpha)}{\left(\alpha - \frac{3}{2}\right)(5\alpha^2 - 12\alpha + 6)}\right]^{3/5} \tag{14b}$$

$$M_{11} = 0; \quad \frac{4\pi^2}{\lambda^2} = f(\alpha)\frac{u_0^2}{\delta_L h_0} \tag{15}$$

Equations (14a) and (14b) are both obtained from the condition $M_{10} = 0$, the difference being that (14b) includes the contribution of the term $\left(\nu \frac{\partial^2 u}{\partial x^2}\right)$ in the original Navier-Stokes relations. Equations (13) to (15) correspond exactly to the results quoted by Kapitza and Massot et al. provided $\beta = 1$ and the appropriate terms are included in the Navier-Stokes equations. Thus, both workers present results that apply only for infinitesimal waves and that are not consistent with the level of approximation. This arises because the effect of the Q_{2i} terms on M_0 is neglected.

Second Approximation

No simple algebraic solution is possible at this level of approximation but Shkadov presents a numerical solution of the problem. However, it is possible to obtain the results of Kapitza's second approximation by neglecting the influence of the Q_{3i} terms on M_{10} and M_{11}. The five conditions obtained by setting the $M_{ij} = 0$ are Equations (13), (14a), and (15) with

$$\phi_{21} = \frac{1}{4} + \frac{(\alpha^2 - \alpha)}{6(5\alpha - 6)} \tag{16}$$

$$= 0.28 \quad \text{for} \quad \alpha = 3 \tag{17}$$

$$\phi_{20} = -g/4N^3 h_0 \,\delta_L \tag{18}$$

These five equations are exactly those quoted by Kapitza in his second approximation. In Equation (16) the dimensionless wave celerity is set equal to three rather than the correct value given by (14a). Thus, the results are neither self consistent nor consistent with the level of approximation.

CONCLUSION

The Shkadov method is the only self consistent solution to falling film flow using the Kapitza approach. The practical limit for handling the algebra is the third approximation which Dukler has shown to be inadequate for describing experimentally observed waveforms. Thus, the method is limited to low Reynolds number flows (< 100), and the possibility of extending its application seems remote.

NOTATION

- c = celerity of waves
- $f(\alpha)$ = $\left(\alpha^2 - \frac{12\alpha}{5} + \frac{6}{5}\right)$ for Massot et al.
  = $\left(\alpha^2 - \frac{11\alpha}{5} + \frac{6}{5}\right)$ for Kapitza
- g = acceleration of gravity
- h = local film thickness
- h_0 = mean film thickness
- M_{ij} = Fourier coefficients defined by Equation (9)
- n = $2\pi h_0/\lambda$
- N = wave number = n/h_0
- Q_i = ith component of Q defined by Equation (7)
- Q_{ij} = coefficients in expansion for Q_i
- R_{k+1} = remainder term in Equation (9)
- Re = Reynolds number
- s = fluid property grouping = $(3\delta_L^3/g\nu^4)^{1/5}$
- t = time
- u = ξ directional velocity component
- $\overline{u}$ = velocity averaged over local film thickness
- u_0 = velocity averaged over h_0
- v = y directional velocity component
- x = downstream coordinate direction
- y = coordinate direction normal to containing wall

Greek Letters

- α = dimensionless celerity = c/u_0
- β = $(gh_0^3/3\nu u_0)^{1/3}$
- δ_L = kinematic surface tension
- λ = wavelength
- ν = kinematic viscosity
- ξ = distance in coordinate system moving with wave
- ρ = wave peak amplitude made dimensionless by h_0
- ρ_L = density
- ϕ = function defined by Equation (3)
- ϕ_{ij} = coefficients in Fourier expansion, Equation (3)

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